Journal of Mathematical Sciences: Advances and Applications
Volume 2, Number 2, 2009, Pages 215-220

A COMPLETE QUASI-ANTIORDER IS THE INTERSECTION OF A COLLECTION OF QUASI-ANTIORDERS

DANIEL ABRAHAM ROMANO

Department of Mathematics and Informatics
Banja Luka University
2, Mladen Stojanovic Street
78000 Banja Luka
Bosnia and Herzegovina
e-mail: bato49@hotmail.com

Abstract

Setting of this paper is Bishop's constructive mathematics. For a relation σ on a set with apartness is called quasi-antiorder if it is consistent and cotransitive. The quasi-antiorder σ is complete if holds $\sigma \cap \sigma^{-1} = \emptyset$. In this paper the following assertion 'A quasi-antiorder is the intersection of a collection of quasi-antiorders.' is given.

1. Introduction and Preliminaries

This short investigation, in Bishop's Constructive Mathematics, is a continuation of the author's forthcoming papers [7]. Bishop's Constructive Mathematics is developed on Constructive logic ([8]) - logic without the

2000 Mathematics Subject Classification: Primary 03F65; Secondary 06B99.

Keywords and phrases: constructive mathematics, set with apartness, anti-order, quasi-antiorder.

This paper is partially supported by the Ministry of science and technology of the Republic of Srpska, Banja Luka, Bosnia and Herzegovina.

Received December 15, 2008

Law, of Excluded Middle $P \vee \neg P$. Let us note that in Constructive Logic the ‘Double Negation Law’ $P \Leftrightarrow \neg\neg P$ does not hold, but the following implication $P \Rightarrow \neg\neg P$ holds even in the Minimal Logic. We have to note that ‘the crazy axiom’ $\neg P \Rightarrow (P \Rightarrow Q)$ is included in the Constructive Logic. In Constructive Logic 'Weak Law of Excluded Middle' $\neg P \vee \neg\neg P$ does not hold, too. It is interesting that in Constructive Logic the following deduction principle $A \vee B,\ A \vdash\ B$ holds, but this is impossible to prove without ‘the crazy axiom’. Bishop's Constructive Mathematics is consistent with the Classical Mathematics.

Relational structure $(X, =, \neq)$, where the relation $\neq$ is a binary relation on X, which satisfies the following properties:

$$\neg(x \neq x),\ x \neq y \Rightarrow y \neq x,\ x \neq z \Rightarrow x \neq y \vee y \neq z,$$

$$x \neq y \wedge y = z \Rightarrow x \neq z,$$

we call *set*. Following Heyting, the relation $\neq$ is called *apartness*. A relation q on X is a *coequality relation* on X if and only if it is consistent, symmetric and cotransitive ([5]-[6]):

$$q \subseteq \neq,\ q^{-1} = q,\ q \subseteq q * q,$$

where “$*$” is *filled product* between relations (see [4]). Let X be a set with an apartness. As in [5], a relation α on X is an *anti-order* on X if and only if

$$\alpha \subseteq \neq,\ \alpha \subseteq \alpha * \alpha,\ \neq \subseteq \alpha \cup \alpha^{-1} \text{ (linearity).}$$

A relation τ on X is a *quasi-antiorder* ([5]) on X if consistent and cotransitive:

$$\tau \subseteq \neq,\ \tau \subseteq \tau * \tau \cdot$$

A (quasi-)antiorder α is *complete* if holds $\alpha \cap \alpha^{-1} = \emptyset$. Let x be an element of X and Y a subset of X. We denote $x \bowtie Y$ if and only if $(\forall a \in Y)(x \neq a)$, and $Y^C = \{x \in S : x \bowtie Y\}$. If τ is a quasi-antiorder on X, then the relation $q = \tau \cup \tau^{-1}$ is a coequality relation on X. Firstly, the relation $q^C = \{(x, y) \in X \times X : (x, y) \bowtie q\}$ is an equivalence on X compatible with q, in the following sense

$$(\forall a, b, c \in X)((a, b) \in q^C \wedge (b, c) \in q \Rightarrow (a, c) \in q).$$

We can construct the factor-set $X / (q^C, q) = \{aq^C : a \in X\}$ with

$$aq^C =_1 bq^C \Leftrightarrow (a, b) \in q^C, aq^C \neq_1 bq^C \Leftrightarrow (a, b) \in q.$$

We can also construct the factor-set $X / q = \{aq : a \in X\}$ with

$$aq =_1 bq \Leftrightarrow (a, b) \bowtie q, aq \neq_1 bq \Leftrightarrow (a, b) \in q.$$

It is easy to check that $X / (q^C, q) \cong X / q$. The mapping $\pi : X \to X / q$, defined by $\pi(a) = aq$ for any $a \in X$, is a strongly extensional surjective mapping. Secondly, note that the relation α^C is an order relation on set $(X, \neg \neq, \neq)$. Following Baroni, if the relation $\neg\alpha$ is an order relation on set $(X, =, \neq)$, when the apartness is tight, $\neg \neq \subseteq =$, then the relation α is called *excise relation* on X.

For a given anti-ordered set $(X, =, \neq, \alpha)$ is essential to know if there exists a coequality q on X such that X / q be an anti-ordered set. This plays an important role for studying the structure of anti-ordered sets. The following question is natural: If $(X, =, \neq, \alpha)$ is an anti-ordered set and q a coequality relation on X, is the set X / q an anti-ordered set? Since, the answer is not affirmative, in general, the following question arises: Is there coequality relation q on X for which X / q is anti-ordered set? The concept of quasi-antiorder relation was studied in [5]. According to [5] and [6], if $(X, =, \neq, \alpha)$ is an anti-ordered set and σ a quasi-antiorder on X, then the relation q on X, defined by $q = \sigma \cup \sigma^{-1}$, is a coequality on X and the set X / q is an ordered set under anti-order Θ defined by $(xq, yq) \in \Theta \Leftrightarrow (x, y) \in \sigma$. So, according to results in [5], each quasi-antiorder σ on an ordered set X under anti-order α induces an coequality relation $q = \sigma \cup \sigma^{-1}$ on X such that X / q is an ordered set under anti-order Θ. In [6] we prove that the converse of this statement also holds. If $(X, =, \neq, \alpha)$ is an anti-ordered set and q a coequality on X and if there exists an anti-order relation Θ_1 on X / q such that $(X / q, =_1, \neq_1, \Theta_1)$ is an ordered set under anti-order Θ_1, then there exists a quasi-antiorder τ on X such that $q = \tau \cup \tau^{-1}$ and $\Theta_1 = \Theta$. So, each coequality q on a set $(X, =, \neq, \alpha)$ such that X / q is an anti-ordered set induces a quasi-antiorder on X.

Anti-orders and quasi-antiorders on set with apartness were investigated by this author in his papers [4], [5] and [6]. What is a connection between complete quasi-antiorder σ and a family $\{\tau : \sigma \subseteq \tau\}$ of quasi-antiorders on X containing σ? - is a question interesting in our understanding of these relations. It is clear that holds $\sigma \subseteq \bigcap\{\tau : \sigma \subseteq \tau\}$. It seems that the following question is natural: Is the following equality $\sigma = \bigcap\{\tau_k : \sigma \subseteq \tau_k\}$ valid for some collection $\{\tau_k : \sigma \subseteq \tau_k\}$. In this paper we give a proof for above equality. So, any complete quasi-antiorder σ on set X is the intersection of a collection of quasi-antiorders on X containing σ.

For the necessary undefined notions and notations, the reader is referred to well-known books [1]-[3], [8] and to papers [4]-[6].

2. The Result

In order to obtain the relationship between coequality and quasi-antiorder on X, the following theorem is essential.

Theorem 2.1 ([5], [6]). *Let $(X, =, \neq, \alpha)$ be an anti-ordered set, q a coequality on X. The following are equivalent:*

(1) *There exists an anti-order θ on factor-set X / q such that $(X / q, =_1, \neq_1, \theta)$ is an ordered set under anti-order θ such that the natural mapping $\pi : X \to X / q$ is a reverse isotone mapping.*

(2) *There exists a quasi-antiorder σ on X, such that $q = \sigma \cup \sigma^{-1}$.*

Secondly, we need the following assertions:

Theorem 2.2 ([7]). *If* α *is a complete anti-order on X, then* α *is the intersection of the anti-orders on X containing* α.

The main result of this paper is the following:

Theorem 2.3. *Every compete quasi-antiorder is the intersection of a collection of quasi-antiorders.*

Proof. Let σ be a complete quasi-antiorder on set X. Then ([5]) the relation θ on $X/(\sigma \cup \sigma^{-1})$, defined by $(aq, bq) \in \theta \Leftrightarrow (a, b) \in \sigma$, is a complete anti-order on $X/(\sigma \cup \sigma^{-1})$. Since, by Theorem 2.2,

$$\theta = \bigcap \{\vartheta : \theta \subseteq \vartheta\},$$

holds, by Theorem 2.1, we have

$$\sigma = \pi^{-1}(\theta) = \bigcap \{\pi^{-1}(\vartheta) : \theta \subseteq \vartheta\},$$

where $\pi^{-1}(\vartheta) = \{(u, v) \in X \times X : (\pi(u), \pi(v)) \in \vartheta\}$, because π is a isotone and reverse isotone mapping. Indeed, if $(x, y) \in \sigma$, then

$$(a(\sigma \cup \sigma^{-1}), b(\sigma \cup \sigma^{-1})) \in \theta = \bigcap \{\vartheta : \theta \subseteq \vartheta\},$$

by Theorem 2.3. Hence, we have $(a(\sigma \cup \sigma^{-1}), b(\sigma \cup \sigma^{-1})) \in \vartheta$ for any anti-order ϑ on factor-set $X/(\sigma \cup \sigma^{-1})$. Thus ([5]), $\pi^{-1}(\vartheta)$ is a quasi-antiorder on X which contains σ. Therefore, we have $\sigma \subseteq \bigcap \{\pi^{-1}(\vartheta) : \theta \subseteq \vartheta\}$. Opposite, let (x, y) be an arbitrary element of $\bigcap \{\pi^{-1}(\vartheta) : \theta \subseteq \vartheta\}$. Then, $(x, y) \in \pi^{-1}(\vartheta)$ for any ϑ of the family $\{\vartheta : \theta \subseteq \vartheta\}$. Thus, $(x, y) \in \theta$ and, finally, $(x, y) \in \sigma$.

References

[1] E. Bishop, Foundations of Constructive Analysis, McGraw-Hill, New York, 1967.

[2] D. S. Bridges and F. Richman, Varieties of Constructive Mathematics, London Mathematical Society Lecture Notes 97, Cambridge University Press, Cambridge, 1987.

[3] R. Mines, F. Richman and W. Ruitenburg, A Course of Constructive Algebra, Springer, New York, 1988.

[4] D. A. Romano, On construction of maximal coequality relation and its applications, In Proceedings of 8th international conference on Logic and Computers Sciences LIRA 97, Novi Sad, September 1-4, 1997, (Editors: R. Tošić and Z. Budimac), Institute of Mathematics, Novi Sad (1997), 225-230.

[5] D. A. Romano, A note on quasi-antiorder in semigroup, Novi Sad J. Math. 37(1) (2007), 3-8.

[6] D. A. Romano, On regular anticongruence in anti-ordered semigroups, Publications de l'Institut Mathmatique 81(95) (2007), 95-102.

[7] D. A. Romano, A complete anti-order is the intersection of family of all antiorders containing it, J. Pure and Appl. Math.: Adv. and Appl. 1(2) (2009), 121-128.

[8] A. S. Troelstra and D. van Dalen, Constructivism in Mathematics, An Introduction, North-Holland, Amsterdam, 1988.

■